Soliton Connection of the sinh-Gordon Equation

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It is demonstrated that the sinh-Gordon equation can be written as covariant exterior derivatives of Lie algebra valued differential forms and, moreover, that these nonlinear differential equations represent a completely integrable system.

Nonlinear evolution equations are usually called integrable if one of the following properties is fulfilled [1]: (1) the initial value problem can be solved exactly with the help of the inverse scattering transform, (2) they have an infinite number of conservation laws, (3) they have an auto Bäcklund transformation or a Bäcklund transformation to a linear equation, (4) besides Lie point vector fields they admit Lie Bäcklund vector fields, (5) they describe pseudospherical surfaces, i.e. surfaces of constant negative Gaussian curvature, (6) they can be written as covariant exterior derivatives of Lie algebra valued differential forms.

It is conjectured that, if property (1) holds, then the properties (2) through (6) also hold. Crampin et al. [2] have introduced the curvature form on bundles to study some geometrical features of soliton equations in field theory, namely the Korteweg de Vries (KdV), the modified KdV, and the sinh-Gordon equations. The basic idea is the relationship between the nonlinear differential equations with soliton solutions and the group SL (2, $\mathbb{R}$). Furthermore, Crampin [3] has deduced the Bäcklund transformations and the equations for the inverse scattering transform from the fact that the SL (2, $\mathbb{R}$) connection has zero curvature. The connection form is explicitly given for some SL (2, $\mathbb{R}$)-valued functions and it is shown that the connection is "pure gauge". In further investigations, I have considered the soliton connection on bundles of the Liouville equation [4], the well known nonlinear Schrödinger equation [5] and nonlinear one-dimensional lattice equations [6]. Moreover, in a further paper [7] a system of partial differential equations, namely the reduced Maxwell-Bloch equations is investigated. These nonlinear equations are the governing equations in the theory of optical self-induced transparency, and they are important in the physics of nonlinear optics.

In the following consideration it is shown that the sinh-Gordon equation in light-cone coordinates can be written as covariant exterior derivatives of Lie algebra valued differential forms. The basic idea is to express the curvature form by the covariant exterior derivative of the 1-form ω on a principal bundle P (M, G) with values in a finite dimensional vector space $\boldsymbol{V}$. The paper is organized as follows: We give a compressed presentation of the theory, cite some formulae and then apply it to the sinh-Gordon equation.

Let us start with the scattering equations

$$\hat{L}\varphi = \varphi_x , \tag{1}$$

where

$$\hat{L} = \begin{pmatrix} \eta & q(x,t) \\ r(x,t) & -\eta \end{pmatrix} . \tag{2}$$

The subscripts indicate partial differentiations. φ is a column vector with transpose $\tilde{\varphi} = (\varphi^1, \varphi^2)$. The time evolution of the functions $\varphi^1(x,t)$ and $\varphi^2(x,t)$ is given by

$$\hat{A}\varphi = \varphi_t , \tag{3}$$

where

$$\hat{A} = \begin{pmatrix} A(x,t;\eta) & B(x,t;\eta) \\ C(x,t;\eta) & -A(x,t;\eta) \end{pmatrix} . \tag{4}$$

The parameter $\eta(x)$ is called eigenvalue of the scattering problem and the quantities $q(x,t)$, $r(x,t)$, $A(x,t;\eta)$, $B(x,t;\eta)$ and $C(x,t;\eta)$ must be given to specify the specific problem under consideration. Rewriting of (1) and (2) in matrix notation we obtain

$$\frac{\partial \varphi^k}{\partial x^j} + \sum_{p=1}^{2} \Gamma^k_{pj} \cdot \varphi^p = 0 , \tag{5}$$

where $j, k = (1, 2)$ and $x^1 = x$, $x^2 = t$. The quantities $\varphi^q(x,t)$ are interpreted as the components of a two-component field on the principle bundle P (M, G). The quantities Γ^k_{pj} are given by the components of the matrix in (2) and (4).

The curvature form Ω is given by the exterior covariant derivative of the 1-form ω on P with values in a finite dimensional vector space $\boldsymbol{V}$. The curvature form can be written as

$$\Omega = \nabla\omega = \mathrm{d}\omega \circ h . \tag{6}$$

Ω is a $\boldsymbol{g}$-valued 2-form and

$$\nabla\omega(X_1, \ldots, X_{p+1}) = \mathrm{d}\omega(hX_1, \ldots, hX_{p+1}) ,$$

where $h\colon T_{\mathrm{p}}(\mathrm{P}(\mathrm{M},\mathrm{G})) \to S_{\mathrm{p}}$ is the projection of the tangential space $T_{\mathrm{p}} = S_{\mathrm{p}} \oplus V_{\mathrm{p}}$ onto its horizontal subspace S_{p}. The space V_{p} of vertical vectors lies tangential to the fibre.

The exterior derivative d is unchanged in its action on forms which take their values in a real vector space $\boldsymbol{V}$. On sections of $\boldsymbol{V} \oplus \Lambda^1(T_{\mathrm{p}}(\mathrm{P}(\mathrm{M},\mathrm{G})))$ we have

$$\mathrm{d}(\omega^j \otimes X_j) = \mathrm{d}\omega^j \otimes X_j , \quad \omega^j \in \Lambda^1(T_{\mathrm{p}}) , \tag{7}$$

where $\{X_k\}^n_{k=1}$ is a basis of $\boldsymbol{V}$. If $\boldsymbol{V}$ is a Lie algebra $\boldsymbol{V} = \boldsymbol{g}$ we can define

$$[\omega^i \otimes X_i, \omega^j \otimes X_j] := (\omega^i \wedge \omega^j) \otimes [X_i, X_j] . \tag{8}$$

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Equation (8) relates R-valued forms to the bracket of $\boldsymbol{g}$-valued forms. The expression (8) is anticommutativ and satisfies the Jacobi identity. Now the curvature form (6) can be expressed. Let $\{X_k\}_{k=1}^{3}$ be a basis of the Lie algebra $\boldsymbol{g} = \mathrm{SL}(2, \mathbb{R})$, then from (6) with (7) and (8) follows

$$\Omega = \sum_{i=1}^{3} \mathrm{d}\omega^i \otimes X_i + \tfrac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} (\omega^i \wedge \omega^j) \otimes [X_i, X_j]. \tag{9}$$

The $\omega^k (k = 1, 2, 3)$ are arbitrary 1-forms and $[X_p, X_q]$ represents the commutator of the quantities X_k. Let

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{10}$$

be a basis of $\boldsymbol{g}$. In view of (5) the 1-forms are expressible as follows:

$$\omega^1 = -(\eta\, \mathrm{d}x + A\, \mathrm{d}t),$$
$$\omega^2 = -(q\, \mathrm{d}x + B\, \mathrm{d}t),$$
$$\omega^3 = -(r\, \mathrm{d}x + C\, \mathrm{d}t). \tag{11}$$

If we take (10) and (11) into account, then the curvature form (9) is given by

$$\begin{aligned} \Omega = {} & (qC - rB - A_x)\, \mathrm{d}x \wedge \mathrm{d}t \otimes X_1, \\ & + (2\eta B - 2qA + q_t - B_x)\, \mathrm{d}x \wedge \mathrm{d}t \otimes X_2, \\ & + (-2\eta C + 2rA + r_t - C_x)\, \mathrm{d}x \wedge \mathrm{d}t \otimes X_3. \end{aligned} \tag{12}$$

We choose

$$A = \frac{1}{4\eta} \cosh u, \quad B = -C = \frac{1}{4\eta} \sinh u \text{ and}$$
$$r = q = -\frac{1}{2} u_x. \tag{13}$$

With (13) we calculate the curvature form (12) and obtain

$$\Omega = \tfrac{1}{2} \mathrm{d}x \wedge \mathrm{d}t \{(\sinh u - u_{xt}) \otimes X_2 + (\sinh u - u_{xt}) \otimes X_3\},$$

or

$$\Omega = \tfrac{1}{2} (\sinh u - u_{xt})\, \mathrm{d}x \wedge \mathrm{d}t \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{14}$$

If $\Omega = 0$, then we conclude from (14) that

$$u_{xt} = \sinh u. \tag{15}$$

Concluding Remarks: It is pointed out that the linear scattering problem for the sinh-Gordon equation may be described in terms of a linear connection on a principal $\mathrm{SL}(2, \mathbb{R})$-bundle. From the condition $\Omega = 0$ we conclude that ω satisfies the structure equation of Maurer-Cartan, $\mathrm{d}\omega + \frac{1}{2}[\omega, \omega] = 0$, and that the connection in $\mathrm{P}(\mathrm{M}, \mathrm{G})$ is flat. Consequently the flat connection has zero curvature. Therefore the sinh-Gordon equation is integrable.

A valuable test for integrability of partial differential equations (pde's) is the Painlevé property. For a field equation considered in the complex domain, Weiss et al. [8] define the Painlevé property as meaning that the solution of a given pde can be represented locally as a single-valued expansion about its movable singular manifold. This means, if u is a solution of a pde then we can write the Painlevé expansion

$$u = \Phi^n \sum_{j=0}^{\infty} u_j \Phi^j, \tag{16}$$

where Φ is the analytic function for which the equation $\Phi = 0$ defines the singular manifold. If we perform the Painlevé test of (15) we obtain that the sinh-Gordon equation in $(1+1)$ dimensions possesses the Painlevé property. This means that the sinh-Gordon equation in $(1+1)$ dimensions in the form $u_{xt} = \sinh u$ or $u_{tt} - u_{xx} = \sinh u$ is integrable. The sinh-Gordon equation $u_{tt} - u_{xx} - u_{yy} = \sinh u$ in $(2+1)$ dimensions or in $(3+1)$ dimensions, however, does not have the Painlevé property. This means that these equations are not integrable. It is conjectured that, if a nonlinear evolution equation (pde) has the Painlevé property then this equation is integrable. On the other hand we cannot conclude, in general, that a pde which is integrable has the Painlevé property.

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